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THOM MODULES

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0. Introduction

This paper presents an algebraic treatment of an analogue of topological K-theory at the level of characteristic classes, as outlined in [2]. It is not what is usually called algebraic K-theory. The idea is to replace the Steenrod Algebra by a fairly general Hopf algebra, topological spaces by algebras over this Hopf algebra, and vector bundles by an algebraic analogue of the cohomology of the Thom space of a vector bundle. We use the term 'Thom module' for this last object. Thom modules have found use by Stephen Mitchell in [6].

1. The abelian group $K(A,H)$

Throughout this paper, all algebras and Hopf algebras will be over a fixed but general ungraded commutative ring with unit R. $\otimes$ will denote tensor product over R. The term 'Hopf algebra' will mean a non-negatively graded, associative, coassociative, cocommutative, connected Hopf algebra over R. If A is a Hopf algebra, the term 'algebra over A' will mean a non-negatively graded, associative, commutative, connected algebra over R which is also an algebra over the Hopf algebra A in the sense of Steenrod [7], i.e. H is also a graded left A-module and we require

(i) (Cartan formula)

$$a \cdot (h_1 \cup h_2) = \sum_i (-1)^{|h_1|\,|a_i''|}(a_i' \cdot h_1) \cup (a_i'' \cdot h_2)$$

whenever $a \in A$, $h_i \in H$, and $\Delta a = \sum_i a_i' \otimes a_i''$. (All elements are assumed homogeneous, $|x|$ denotes the grade of x, and we write $\cup$ for the product operation in H as well as for left H-module actions below. Without explicit mention, a_i' and a_i'' will always be as above and

$$\sum_i a_i' \otimes a_i'' = a \otimes 1 + 1 \otimes a + \sum_{0<|a_i'|<|a|} a_i' \otimes a_i''$$

when $|a|>0$.)

(ii) $a \cdot 1 = 0$ whenever $|a| > 0$.

We consider A-H modules M (or equivalently $H \odot A$-modules where $H \odot A$ is the semi-tensor product of H and A as in [3]), i.e. M is both a graded left A-module and a graded left H-module such that the Cartan formula (i) holds with h_2 replaced by an element of M.

Definition 1.1. An A-H *Thom module* is an A-H module which, as a module over H, is free on one zero-dimensional generator.

The motivating example is when $A = \mathscr{A}_2 = \text{mod}\,2$ Steenrod algebra, $H = H^*(X; \mathbb{Z}/2)$ where X is a topological space, and $M = \tilde{H}^{*+n}(T(\xi); \mathbb{Z}/2)$ where ξ is a real n-plane bundle over X and $T(\xi)$ is the Thom space of ξ.

If M_1, M_2 are A-H modules, so is $M_1 \otimes_H M_2$ with A-action given by

$$a \cdot (m_1 \otimes m_2) = \sum_i (-1)^{|a_i''|\,|m_1|} a_i' \cdot m_1 \otimes a_i'' \cdot m_2, \quad m_i \in M_i,\ a \in A.$$

Coassociativity and cocommutativity of A ensure that the natural H-isomorphisms

$$\left(M_1 \underset{H}{\otimes} M_2\right) \underset{H}{\otimes} M_3 \cong M_1 \underset{H}{\otimes} \left(M_2 \underset{H}{\otimes} M_3\right), \qquad M_1 \underset{H}{\otimes} M_2 \cong M_2 \underset{H}{\otimes} M_1$$

for A-H modules M_i are also A-maps. If M_1, M_2 are A-H Thom modules, so is $M_1 \otimes_H M_2$.

Definition 1.2. Let H be an algebra over the Hopf algebra A. The *A-characteristic K-group of H*, denoted $K(A, H)$, is the set of all A-H isomorphism classes of A-H Thom modules.

It follows from the foregoing that $\otimes_H$ induces an operation on $K(A, H)$ giving $K(A, H)$ the structure of a commutative semigroup. There is a unit element for this operation, namely the class of H. We will see below that inverses always exist, giving $K(A,, H)$ the structure of an abelian group. We write $+$ for the above operation in $K(A, H)$, and $[M]$ for the element of $K(A, H)$ represented by the A-H Thom module M.

In the motivating example, $\otimes_H$ corresponds to Whitney sum of vector bundles. Thus, for finite complexes X, there is an evident natural homomorphism $\tilde{K}O(X) \to K(\mathscr{A}_2, H^*(X; \mathbb{Z}/2))$ which, in general, is neither injective nor surjective.

Given an A-H Thom module M, the A-action on M is described as follows: If $U \in M^\circ$ generates M as an H-module, then for each $a \in A$, there exists a unique $w(a, M) \in H^{|a|}$ such that $a \cdot U = w(a, M) \cup U$.

Definition 1.3. The function $w(\cdot, M) : A \to H$ is called the *Stiefel–Whitney map* of the A-H Thom module M.

Thus $w(\cdot, M)$ is a grade-preserving R-homomorphism and determines M up to A-H isomorphism. Note that $w(1, M) = 1$.

In the motivating example above, $w(\mathrm{Sq}^i, \tilde{H}^{*+n}(T(\xi); \mathbb{Z}/2)) = w_i(\xi)$, the ith Stiefel–Whitney class of the n-plane bundle ξ [5].

Proposition 1.4 (Whitney Product Formula). *If M_1, M_2 are A-H Thom modules, then*

$$w\left(a, M_1 \underset{H}{\otimes} M_2\right) = \sum_i w(a_i', M_1) \cup w(a_i'', M_2).$$

The proof is easy.

Given an A-H Thom module M, it is easily seen, by induction on grade, that there exists a unique grade-preserving function $\bar{w}(\cdot, M) : A \to H$ satisfying $\bar{w}(1, M) = 1$ and

$$\sum_i w(a_i', M) \cup \bar{w}(a_i'', M) = 0 \quad \text{for } |a| > 0.$$

Definition 1.5. $\bar{w}(\cdot, M)$ is called the *dual Stiefel–Whitney map* of M.

By the Whitney Product Formula, if there exists an inverse to $[M]$ in $K(A, H)$, i.e. an A-H Thom module $\bar{M}$ such that $M \otimes_H \bar{M}$ is A-H isomorphic to H, then necessarily $w(\cdot, \bar{M}) = \bar{w}(\cdot, M)$. Thus to prove that $K(A, H)$ is a group, it remains to show that for each A-H Thom module M, $\bar{w}(\cdot, M)$ is the Stiefel–Whitney map of an A-H Thom module. We proceed to obtain a criterion (the *Composition Formula*) for a general map $w : A \to H$ to be the Stiefel–Whitney map of an A-H Thom module, and then prove that $\bar{w}(\cdot, M)$ satisfies this criterion for every A-H Thom module M.

Theorem 1.6 (Composition Formula). *Let $w : A \to H$ be a graded R-homomorphism satisfying $w(1) = 1$. Then w is the Stiefel–Whitney map of an A-H Thom module if and only if for all $a, b \in A$,*

$$w(a \cdot b) = \sum_i (-1)^{|a_i''|\,|b|} [a_i' \cdot w(b)] \cup w(a_i'').$$

Proof. If $w = w(\cdot, M)$ for an A-H Thom module M, generated by $U \in M^\circ$ as an H-module, the composition formula expresses the condition $(a\, b) \cdot U = a \cdot (b \cdot U)$.

Conversely, suppose w satisfies the composition formula. Let M be the free left H-module on one generator $U \in M^\circ$. For $a \in A$, $h \in H$, define

$$a \cdot (h \cup U) = \sum_i (-1)^{|h|\,|a_i''|} (a_i' \cdot h) \cup w(a_i'') \cup U.$$

We wish to verify that this defines an A-action on M, giving M the structure of an A-H Thom module with $w(\cdot, M) = w$. The only non-trivial point to be checked is that

$$(a\, b) \cdot (h \cup U) = a \cdot [b \cdot (h \cup U)] \quad \text{whenever } a, b \in A,\ h \in H.$$

Whenever $G_1, \ldots, G_n$ are graded R-modules and σ is a permutation of $\{1, \ldots, n\}$, let

$$T_\sigma : G_1 \otimes \cdots \otimes G_n \to G_{\sigma(1)} \otimes \cdots \otimes G_{\sigma(n)}$$

denote the graded permutation map. Let $\mu_A : A \otimes A \to A$ and $\mu_H : H \otimes H \to H$ denote the respective multiplication maps, and $\varphi : A \otimes H \to H$ the A-action. Let $\alpha : A \otimes H \to H$ denote the composition

$$A \otimes H \xrightarrow{\Delta \otimes 1} A \otimes A \otimes H \xrightarrow{T_{(23)}} A \otimes H \otimes A \xrightarrow{\varphi \otimes w} H \otimes H \xrightarrow{\mu_H} H.$$

Then $a \cdot (h \cup U) = \alpha(a \otimes h) \cup U$, and the condition to be checked is commutativity of

(1)
$$\begin{tikzcd}
A \otimes A \otimes H \arrow[r, "\mu_A \otimes 1"] \arrow[d, "1 \otimes \alpha"'] & A \otimes H \arrow[d, "\alpha"] \\
A \otimes H \arrow[r, "\alpha"'] & H
\end{tikzcd}$$

The composition formula asserts commutativity of

(2)
$$\begin{tikzcd}
A \otimes A \arrow[r, "\Delta \otimes 1"] \arrow[dd, "\mu_A"] & A \otimes A \otimes A \arrow[r, "T_{(23)}"] & A \otimes A \otimes A \arrow[r, "1 \otimes w \otimes w"] & A \otimes H \otimes H \arrow[d, "\varphi \otimes 1"] \\
 & & & H \otimes H \arrow[d, "\mu_H"] \\
A \arrow[rrr, "w"] & & & H
\end{tikzcd}$$

A direct check on elements yields commutativity of

(3)
$$\begin{tikzcd}
A \otimes A \otimes H \arrow[r, "\Delta \otimes \Delta \otimes 1"] \arrow[d, "\alpha(1 \otimes \alpha)"] & A \otimes A \otimes A \otimes A \otimes H \arrow[r, "T_{(253)}"] & A \otimes A \otimes H \otimes A \otimes A \arrow[d, "1 \otimes \varphi \otimes w \otimes w"] \\
H & H \otimes H \arrow[l, "\mu_H"] & A \otimes H \otimes H \arrow[l, "\varphi \otimes 1"] & A \otimes H \otimes H \otimes H \arrow[l, "1 \otimes \mu_H \otimes 1"]
\end{tikzcd}$$

We have the commutative diagram

$$
\begin{tikzcd}
A\otimes H\otimes H\otimes H \arrow[r, "\Delta\otimes 1\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu_H\otimes 1"'] & A\otimes A\otimes H\otimes H\otimes H \arrow[r, "T_{(23)}"] & A\otimes H\otimes A\otimes H\otimes H \arrow[d, "\varphi\otimes\varphi\otimes 1"] \\
A\otimes H\otimes H \arrow[r, "\varphi\otimes 1"'] & H\otimes H \arrow[d, "\mu_H"'] & H\otimes H\otimes H \arrow[l, "\mu_H\otimes 1"] \arrow[d, "1\otimes\mu_H"] \\
& H & H\otimes H \arrow[l, "\mu_H"']
\end{tikzcd}
\tag{4}
$$

the top polygon commuting by the Cartan formula, and the bottom square by associativity of H. A direct check on elements yields commutativity of

$$
\begin{tikzcd}
A\otimes A\otimes A\otimes A\otimes H \arrow[r, "\Delta\otimes 1\otimes 1\otimes 1\otimes 1"] \arrow[dd, "T_{(253)}"'] & A\otimes A\otimes A\otimes A\otimes A\otimes H \arrow[r, "T_{(64)}"] & A\otimes A\otimes A\otimes H\otimes A\otimes A \arrow[d, "1\otimes 1\otimes\varphi\otimes 1\otimes 1"] \\
& & A\otimes A\otimes H\otimes A\otimes A \arrow[d, "T_{(23)}"] \\
A\otimes A\otimes H\otimes A\otimes A \arrow[dd, "1\otimes\varphi\otimes w\otimes w"'] & & A\otimes H\otimes A\otimes A\otimes A \arrow[d, "\varphi\otimes 1\otimes 1\otimes 1"] \\
& & H\otimes A\otimes A\otimes A \arrow[d, "1\otimes 1\otimes w\otimes w"] \\
A\otimes H\otimes H\otimes H \arrow[dd, "\Delta\otimes 1\otimes 1\otimes 1"'] & & H\otimes A\otimes H\otimes H \arrow[d, "1\otimes\varphi\otimes 1"] \\
& & H\otimes H\otimes H \\
A\otimes A\otimes H\otimes H\otimes H \arrow[r, "T_{(23)}"'] & A\otimes H\otimes A\otimes H\otimes H \arrow[ur, "\varphi\otimes\varphi\otimes 1"'] &
\end{tikzcd}
\tag{5}
$$

We have the diagram

$$\begin{tikzcd}
A\otimes A\otimes H \arrow[r, "\Delta\otimes\Delta\otimes 1"] \arrow[d, "\Delta\otimes\Delta\otimes 1"] & A\otimes A\otimes A\otimes A\otimes H \arrow[d, "\Delta\otimes 1\otimes 1\otimes 1\otimes 1"] & \\
A\otimes A\otimes A\otimes A\otimes H \arrow[r, "1\otimes\Delta\otimes 1\otimes 1\otimes 1"'] \arrow[d, "T_{(2453)}"] & A\otimes A\otimes A\otimes A\otimes A\otimes H \arrow[r, "T_{(364)}"'] & A\otimes A\otimes A\otimes H\otimes A\otimes A \arrow[d, "1\otimes 1\otimes\varphi\otimes 1\otimes 1"] \\
A\otimes A\otimes H\otimes A\otimes A \arrow[d, "1\otimes\varphi\otimes 1\otimes 1"] & & A\otimes A\otimes H\otimes A\otimes A \arrow[d, "T_{(23)}"] \\
A\otimes H\otimes A\otimes A \arrow[d, "\varphi\otimes 1\otimes 1"] & & A\otimes H\otimes A\otimes A\otimes A \arrow[d, "\varphi\otimes 1\otimes 1\otimes 1"] \\
H\otimes A\otimes A \arrow[r, "1\otimes\Delta\otimes 1"'] & H\otimes A\otimes A\otimes A \arrow[r, "T_{(14)}"'] & H\otimes A\otimes A\otimes A
\end{tikzcd}$$

(6)

where the top square commutes by coassociativity of A, and the bottom polygon commutes by a direct check on elements.

Diagrams (3), (4), (5), (6) and the diagram obtained from (2) by tensoring on the left with the identity map on H, yield the commutative diagram

$$\begin{tikzcd}
A\otimes A\otimes H \arrow[r, "\Delta\otimes\Delta\otimes 1"] \arrow[d, "\alpha(1\otimes\alpha)"] & A\otimes A\otimes A\otimes A\otimes H \arrow[rrr, "T_{(2453)}"] & & & A\otimes A\otimes H\otimes A\otimes A \arrow[d, "1\otimes\varphi\otimes 1\otimes 1"] \\
H & H\otimes H \arrow[l, "\mu_H"] & H\otimes A \arrow[l, "1\otimes w"] & H\otimes A\otimes A \arrow[l, "1\otimes\mu_A"] & A\otimes H\otimes A\otimes A \arrow[l, "\varphi\otimes 1\otimes 1"]
\end{tikzcd}$$

(7)

We have the diagram

$$
(8)\qquad
\begin{tikzcd}
A\otimes H\otimes A\otimes A \arrow[dd, "\varphi\otimes 1\otimes 1"] & A\otimes A\otimes H\otimes A\otimes A \arrow[l, "1\otimes\varphi\otimes 1\otimes 1"'] \arrow[r, "\mu_A\otimes 1\otimes\mu_A"] \arrow[d, "1\otimes\varphi\otimes\mu_A"] & A\otimes H\otimes A \arrow[d, "\varphi\otimes w"] \\
& A\otimes H\otimes A \arrow[r, "\varphi\otimes"'] \arrow[d, "\varphi\otimes 1"] & H\otimes H \arrow[dd, "\mu_H"] \\
H\otimes A\otimes A \arrow[r, "1\otimes\mu_A"'] & H\otimes A \arrow[d, "1\otimes w"] & \\
& H\otimes H \arrow[r, "\mu_H"] & H
\end{tikzcd}
$$

where commutativity of the two pentagons is immediate, and commutativity of the square follows from the fact that $a\cdot(b\cdot h)=(ab)\cdot h$ for all $a,b\in A$, $h\in H$.

Diagrams (7) and (8) yield commutativity of

$$
(9)\qquad
\begin{tikzcd}
A\otimes A\otimes H \arrow[r, "\Delta\otimes\Delta\otimes 1"] \arrow[d, "\alpha(1\otimes\alpha)"] & A\otimes A\otimes A\otimes A\otimes H \arrow[r, "T_{(2453)}"] & A\otimes A\otimes H\otimes A\otimes A \arrow[d, "\mu_A\otimes 1\otimes\mu_A"] \\
H & H\otimes H \arrow[l, "\mu_H"] & A\otimes H\otimes A \arrow[l, "\varphi\otimes w"]
\end{tikzcd}
$$

Finally, a direct check of elements, using the fact that Δ is an algebra homomorphism. yields commutativity of the diagram obtained from (9) on replacing $\alpha(1\otimes\alpha)$ by $\alpha(\mu_A\otimes 1)$, completing the proof.

Theorem 1.7. *If H is an algebra over the Hopf algebra A, $K(A,H)$ is an abelian group under the operation induced by $\otimes_H$ on A-H Thom modules.*

Proof. Let M be an A-H Thom module and write $w=w(\cdot,M)$, $\bar{w}=\bar{w}(\cdot,M)$. It remains to show that $\bar{w}$ satisfies the composition formula. We proceed to show that the composition formula holds for $\bar{w}(ab)$ by induction on $|a|+|b|$. Trivially, the composition formula holds for $\bar{w}(ab)$ when either $|a|=0$ or $|b|=0$. Suppose $|a|>0$, $|b|>0$, and that the composition formula holds for $\bar{w}(a'b')$ whenever $|a'|+|b'|<|a|+|b|$. Write

$$\Delta a=\sum_i a_i'\otimes a_i'',\qquad \Delta b=\sum_j b_j'\otimes b_j''.$$

Then

$$\Delta(ab)=\sum_{i,j}(-1)^{|a_i''|\,|b_j'|}a_i'b_j'\otimes a_i''b_j''.$$

Thus by definition of $\bar{w}$

(1) $$\sum_j w(b_j')\cup\bar{w}(b_j'')=0,\quad \text{and}$$

(2) $$\sum_{i,j}(-1)^{|a_i''|\,|b_j'|}w(a_i'b_j')\cup\bar{w}(a_i''b_j'')=0.$$

By the inductive hypothesis we can replace each $\bar{w}(a_i''b_j'')$ in (2) for which $|a_i''|+|b_j''|<|a|+|b|$ by the composition formula expression for it. By the inductive definition of $\bar{w}$, it suffices to show that (2) holds when the $\bar{w}(ab)$ term on the left is also replaced by the composition formula expression for it. Write

$$\Delta a_i'=\sum_{p(i)}c_{p(i)}'\otimes c_{p(i)}'',\qquad \Delta a_i''=\sum_{q(i)}d_{q(i)}'\otimes d_{q(i)}''.$$

Replacing each $w(a_i'b_j')$ and $\bar{w}(a_i''b_j'')$ in (2) by the composition formula expression for it, the left-hand side of (2) becomes the image of

(3) $$\sum_{i,j,p(i),q(i)}c_{p(i)}'\otimes c_{p(i)}''\otimes d_{q(i)}'\otimes d_{q(i)}''\otimes b_j'\otimes b_j''$$

under the composition

(4)
$$\begin{array}{l}
A\otimes A\otimes A\otimes A\otimes A\otimes A\\
\quad\Big\downarrow T_{(23465)}\\
A\otimes A\otimes A\otimes A\otimes A\otimes A\\
\quad\Big\downarrow 1\otimes w\otimes w\otimes 1\otimes\bar{w}\otimes\bar{w}\\
A\otimes H\otimes H\otimes A\otimes H\otimes H\xrightarrow{\varphi\otimes 1\otimes\varphi\otimes 1}H\otimes H\otimes H\otimes H\xrightarrow{\text{mult.}}H.
\end{array}$$

By coassociativity and cocommutativity of A, the expression (3) equals

(5) $$\sum_{i,j,p(i),q(i)}(-1)^{|c_{p(i)}''|\,|d_{q(i)}'|}c_{p(i)}'\otimes d_{q(i)}'\otimes c_{p(i)}''\otimes d_{q(i)}''\otimes b_j'\otimes b_j''.$$

Evaluating the composition (4) on the expression in (5), a straightforward computation using the associativity and commutativity of H and the Cartan formula yields

$$\sum_i(-1)^{|a_i''|\,|b|}a_i'\cdot\left[\sum_j w(b_j')\cup\bar{w}(b_j'')\right]\cup\left[\sum_{q(i)}w(d_{q(i)}')\cup\bar{w}(d_{q(i)}'')\right]$$

which is 0 by (1), completing the proof.

2. Functoriality and representability of $K(A,H)$

For a fixed non-negatively graded, associative, commutative, connected algebra H over the ground ring R, we can form the category of Hopf algebras under H. An object of this category is a Hopf algebra A over R, together with a given action of A on H making H an algebra over the Hopf algebra A. A morphism $f: A \to B$ in this category is a unit and grade-preserving homomorphism of Hopf algebras over R such that $a \cdot h = f(a) \cdot h$ for all $a \in A$, $h \in H$. If $f: A \to B$ is a morphism of Hopf algebras under H, and M is a B-H Thom module, we obtain an A-H Thom module f^*H by taking $f^*M = M$ as a left H-module, and imposing a left A-action via f and the given left B-action. In terms of Stiefel–Whitney maps, $w(\cdot, f^*M) = w(\cdot, M) \circ f$. We obtain a map $f^*: K(B,H) \to K(A,H)$, easily seen to be a group homomorphism.

Dually, for a fixed Hopf algebra A, we can form the usual category of algebras over A. If $g: H \to J$ is a morphism of algebras over A and M is an A-H Thom module, we obtain an A-J Thom module $g_*M = J \otimes_H M$ by regarding J as an A-H module via g. In terms of Stiefel–Whitney maps, $w(\cdot, g_*M) = g \circ w(\cdot, M)$. We obtain a map $g_*: K(A,H) \to K(A,J)$, easily seen to be a group homomorphism.

The following proposition is immediate.

Proposition 2.1. *For fixed H as above, $K(\cdot,\ H)$ is a contravariant functor from the category of Hopf algebras under H to the category of abelian groups.*

For a fixed Hopf algebra A, $K(A, \cdot)$ is a covariant functor from the category of algebras over A to the category of abelian groups.

For a fixed Hopf algebra A we proceed to construct a representing object for the functor $K(A, \cdot)$, i.e. an algebra H_A over the Hopf algebra A together with an A-H_A Thom module M_A such that for each A-H Thom module M, there is a unique morphism $f_M: H_A \to H$ of algebras over A such that M is A-H isomorphic to $(f_M)_*(M_A)$.

Definition 2.2. Let A be a Hopf algebra. For each homogeneous element $a \in A$ of positive grade, associate an abstract symbol $w(a)$. Let L_A denote the free graded left A-module on the graded set $\{w(a) \mid a \in A,\ |a| > 0\}$ where $|w(a)| = |a|$, and T_A the tensor algebra over the ring R on L_A. The left A-action on L_A extends uniquely to a left A-action on T_A, making T_A a non-commutative algebra over the Hopf algebra A (the free associative algebra on L_A over the Hopf algebra A). Let I_A denote the smallest ideal in T_A which is closed under the action on A, and which contains all elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x, \quad w(a+b) - w(a) - w(b), \quad w(ka) - k w(a),$$

and

$$w(ab) - \sum_i (-1)^{|a_i''||b|} [a_i' \cdot w(b)] \otimes w(a_i'')$$

where $x, y \in L_A$, a, b are homogeneous elements in A of positive grade, and $k \in R$. The *classifying algebra* H_A of A is defined to be T_A/I_A.

H_A is an algebra over A. Write $w(a) \in H_A$ for the image of $w(a)$ in T_A under the projection. By definition of I_A, the map $w: A \to H_A$ which sends 1 to 1 and a to $w(a)$, $|a|>0$, satisfies the composition formula, and hence is the Stiefel–Whitney map of an A-H_A Thom module M_A. Moreover, if H is an arbitrary algebra over the Hopf algebra A, and M an A-H Thom module, there is a unique homomorphism of algebras over $A, f_M: H_A \to H$, such that $f_M(w(a)) = w(a, M)$. Thus M is A-H isomorphic to $(f_M)_*(M_A)$.

Definition 2.3. M_A is called the *classifying Thom module* for A. f_M is called the *classifying map* of the A-H Thom module M.

If H and J are algebras over the Hopf algebra A, so is $H \otimes J$ with A-action given by

$$a \cdot (h \otimes j) = \sum_i (-1)^{|a_i''||h|}(a_i' \cdot h) \otimes (a_i'' \cdot j).$$

Moreover, if M is an A-H module, N an A-J module, $M \otimes N$ is an A-$H \otimes J$ module with A-action given by

$$a \cdot (m \otimes n) = \sum_i (-1)^{|a_i''||m|}(a_i' \cdot m) \otimes (a_i'' \cdot n).$$

In particular, if M is an A-H Thom module, N an A-J Thom module, then $M \otimes N$ is an A-$H \otimes J$ Thom module.

Proposition 2.4. *If M is an A-H Thom module, N an A-J Thom module, then*

$$w(a, M \otimes N) = \sum_i w(a_i', M) \otimes w(a_i'', N).$$

The proof is immediate.

In particular, for any Hopf algebra $A, M_A \otimes M_A$ is an A-$H_A \otimes H_A$ Thom module.

Proposition 2.5. *For any Hopf algebra A, H_A is a Hopf algebra with diagonal $\Delta : H_A \to H_A \otimes H_A$ the classifying map for the A-$H_A \otimes H_A$ Thom module $M_A \otimes M_A$. Moreover, $w: A \to H_A$ is a morphism of coalgebras over R.*

The proof is straightforward, using the universality of M_A.

Thus for any algebra H over the Hopf algebra A, the set of unit and grade-preserving A-algebra homomorphisms $\mathrm{Alg}_A(H_A, H)$ becomes an abelian group with operations as follows: If $f, g \in \mathrm{Alg}_A(H_A, H), f+g$ is the composition

$$H_A \xrightarrow{\Delta} H_A \otimes H_A \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu_H} H$$

and $-f$ is the composition

$$H_A \xrightarrow{\chi} H_A \xrightarrow{f} H$$

where χ is the canonical conjugation of the Hopf algebra H_A [4].

Proposition 2.6. *Let H be an algebra over the Hopf algebra A. Then the map $K(A, H) \to \mathrm{Alg}_A(H_A, H)$ sending $[M]$ to f_M is a natural isomorphism of abelian groups.*

The proof is straightforward.

Remark 2.7. In the above theory we have not attempted to formulate the analogue of unstable algebras over the Steenrod algebra [7]. In particular, in the case of the mod 2 Steenrod algebra, since $H_{\mathscr{A}_2}$ is universal for all algebras (not just unstable ones) over $\mathscr{A}_2$, $H_{\mathscr{A}_2} \neq H^*(\mathrm{BO}; \mathbb{Z}/2)$. In fact, $\mathrm{Sq}^2 w(\mathrm{Sq}^1) \neq 0$ in $H_{\mathscr{A}_2}$.

If $f: A \to B$ is a homomorphism of Hopf algebras, and H is an algebra over B, H becomes an algebra over A with action $a \cdot h = f(a) \cdot h$, and f is a morphism of Hopf algebras under H. In particular H_B becomes an algebra over A. Let $Hf: H_A \to H_B$ denote the classifying map for the A-H_B Thom module f^*M_B. Explicitly,

$$(Hf)(w(a)) = w(f(a)) \quad \text{for all } a \in A.$$

Proposition 2.8. *If $f: A \to B$ is a homomorphism of Hopf algebras, then $Hf: H_A \to H_B$ is a homomorphism of Hopf algebras, and the classifying algebra becomes a covariant functor from the category of Hopf algebras to the category of commutative Hopf algebras.*

The proof is straightforward.

3. Examples

We show how to calculate $K(A, H)$ when $A = R[x_1, \ldots, x_m] \otimes E(y_1, \ldots, y_n)$, the tensor product of a polynomial algebra over R on even-dimensional generators x_i, and an exterior algebra over R on odd-dimensional generators y_i. Theorem 3.1 below reduces the computation to the cases of polynomial and exterior algebras on a single generator.

Suppose A and B are Hopf algebras. We have Hopf algebra homomorphisms $i_A: A \to A \otimes B$, $i_B: B \to A \otimes B$, $p_A: A \otimes B \to A$, $p_B: A \otimes B \to B$ given by

$$i_A(a) = a \otimes 1, \qquad i_B(b) = 1 \otimes b,$$

$$p_A(a \otimes b) = \varepsilon_B(b)a, \qquad p_B(a \otimes b) = \varepsilon_A(a)b$$

where $\varepsilon_A, \varepsilon_B$ are the respective augmentations. If H is an algebra over $A \otimes B$, H becomes an algebra over A via i_A and over B via i_B.

Theorem 3.1. *Let A and B be Hopf algebras, and suppose H is an algebra over $A \otimes B$. Let $\alpha: K(A \otimes B, H) \to K(A, H) \oplus K(B, H)$ and $\beta: K(A, H) \oplus K(B, H) \to K(A \otimes B, H)$ be given by*

$$\alpha[M] = (i_A^*[M], i_B^*[M]), \qquad \beta([M],[N]) = p_A^*[M] + p_B^*[N].$$

Then α and β are isomorphisms, inverse to one another.

Proof.

$$\alpha\beta([M],[N]) = (i_A^* p_A^*[M] + i_A^* p_B^*[N], i_B^* p_A^*[M] + i_B^* p_B^*[N]).$$

Since $p_A i_A = 1_A$, $i_A^* p_A^*[M] = [M]$. Similarly $i_B^* p_B^*[N] = [N]$. since $p_B i_A$ and $p_A i_B$ both factor through the Hopf algebra R and $K(R, H) = 0$, $i_A^* p_B^*[M] = 0 = i_B^* p_A^*[N]$, and so $\alpha\beta$ is the identity.

$$\beta\alpha[M] = p_A^* i_A^*[M] + p_B^* i_B^*[M] = \left[(i_A p_A)^* M \underset{H}{\otimes} (i_B p_B)^* M\right]$$

and so it suffices to check

$$(1) \qquad w(\cdot, M) = w(\cdot, (i_A p_A)^* M \underset{H}{\otimes} (i_B p_B)^* M)$$

for all $A \otimes B$-H Thom modules M. Since for all $a \in A$, $b \in B$, $a \otimes b = (a \otimes 1)(1 \otimes b)$ in $A \otimes B$, it follows from the composition formula that any Stiefel–Whitney map for an $A \otimes B$-H Thom module is determined by its values on elements of the form $a \otimes 1$ and $1 \otimes b$. Since $\Delta(a \otimes 1) = \sum_i (a_i' \otimes 1) \otimes (a_i'' \otimes 1)$, we have, by the Whitney product formula,

$$\begin{aligned} w(a \otimes 1, (i_A p_A)^* M \underset{H}{\otimes} (i_B p_B)^* M) &= \sum_i w(a_i' \otimes 1, (i_A p_A)^* M) \cup w(a_i'' \otimes 1, (i_B p_B)^* M) \\ &= \sum_i w(i_A p_A(a_i' \otimes 1), M) \cup w(i_B p_B(a_i'' \otimes 1), M) \\ &= \sum_i w(a_i' \otimes 1, M) \cup w(i_B p_B(a_i'' \otimes 1), M). \end{aligned}$$

Since $i_B p_B(a_i'' \otimes 1) = 0$ unless $|a_i''| = 0$, the only non-zero contribution to this last sum occurs when $a_i' = a$ and $a_i'' = 1$, and so (1) holds on elements of the form $a \otimes 1$. Similarly (1) holds on elements of the form $1 \otimes b$, completing the proof.

Example 3.2. $A = E(x)$, the exterior algebra over R on an odd-dimensional generator x. For any algebra H over $E(x)$, the function $w(x, \cdot): K(E(x), H) \to H^{|x|}$ is injective, and is an additive homomorphism since x is primitive. The composition formula yields that an element $h \in H^{|x|}$ is in the image of $w(x, \cdot)$ if and only if $x \cdot h = h^2$. Thus $w(x, \cdot)$ is an isomorphism of $K(E(x), H)$ onto the the additive group

$\{h \in H^{|x|} \mid x \cdot h = h^2\}$. The classifying algebra $H_{E(x)}$ is $R[w(x)]/(2w(x)^2)$ with $E(x)$-action given by $x \cdot w(x) = w(x)^2$.

Example 3.3. $A = R[x]$, the polynomial algebra over R on a positive even-dimensional generator x. As in Example 3.2, the function $w(x, \cdot): K(R[x], H) \to H^{|x|}$ is an injective additive homomorphism. It is easily seen that there is no restriction on the image and so $w(x, \cdot)$ is an isomorphism of $K(R[x], H)$ onto $H^{|x|}$ for every algebra H over $R[x]$. The classifying algebra $H_{R[x]}$ is $R[w(x), x \cdot w(x), x^2 \cdot w(x), \ldots]$.

The following is immediate from Proposition 2.6.

Proposition 3.4. *Let A be a Hopf algebra and suppose $\{H_\alpha\}$ is an inverse system of algebras over A. Then $K(A, \varprojlim_\alpha H_\alpha)$ is naturally isomorphic to $\varprojlim_\alpha K(A, H_\alpha)$.*

Example 3.5. Let H be an arbitrary algebra over the mod 2 Steenrod algebra $\mathscr{A}_2$, and M a non-trivial $\mathscr{A}_2$-H Thom module. If r is the smallest positive integer such that $w(\mathrm{Sq}^r, M) \neq 0$, r must be a power of 2. This follows from the composition formula and the decomposability of Sq^i in $\mathscr{A}_2$ if i is not a power of 2 [1].

Let λ denote the canonical real line bundle over real projective n-space $\mathbb{R}P^n$, and let $L = \tilde{H}^{*+1}(T(\lambda); \mathbb{Z}/2)$. If M is an arbitrary $\mathscr{A}_2$-$H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ Thom module, and r the smallest positive integer such that $w(\mathrm{Sq}^r, M) \neq 0$, it follows from the Whitney product formula that either

$$\underbrace{L \otimes \cdots \otimes L}_{r} \otimes M$$

(tensor product over $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$) is a trivial $\mathscr{A}_2$-$H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ Thom module, or the smallest i for which

$$w(\mathrm{Sq}^i, \underbrace{L \otimes \cdots \otimes L}_{r} \otimes M) \neq 0$$

is strictly larger than r. By an induction on $n - r$, it follows that there exists a positive q such that

$$\underbrace{L \otimes \cdots \otimes L}_{q} \otimes M$$

is trivial, and consequently $[L]$ generates $K(\mathscr{A}_2, H^*(\mathbb{R}P^n; \mathbb{Z}/2))$. Since $w(\mathrm{Sq}^1, L)$ generates $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ and $w(\mathrm{Sq}^i, L) = 0$ for $i > 1$, it follows easily that

$$K(\mathscr{A}_2, H^*(\mathbb{R}P^n; \mathbb{Z}/2)) \cong \mathbb{Z}/2^k$$

with generator $[L]$ where 2^k is the smallest power of 2 which exceeds n. Thus, from Proposition 3.4, it follows that

$$K(\mathscr{A}_2, H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)) \cong \varprojlim_k \mathbb{Z}/2^k.$$

4. Wu classes and the Wu formula

Classically the Wu classes of a closed n-manifold X are defined, using Poincaré duality, to describe the action of the Sq^i into $H^n(X; \mathbb{Z}/2)$ [8]. The Wu formula expresses the Stiefel–Whitney classes of X as linear combinations of the Sq^i on the Wu classes, from which it is possible to inductively determine the Wu classes from knowledge of the Stiefel–Whitney classes and the action of the Steenrod algebra. We take the Wu formula as the basis for defining the Wu classes of an arbitrary A-H Thom module, and then prove that under certain circumstances (analogous to the condition that the top cohomology of the Thom space of the normal bundle be spherically generated) cupping with these Wu classes gives the action of A into H^n for appropriate n. Poincaré duality is not required for this treatment. Indeed, we give an example of a topological situation where cupping with the Wu classes of a vector bundle gives the action of the Steenrod algebra on the cohomology of the base space into an appropriate dimension, yet the base space is not a Poincaré duality space.

Definition 4.1. Let H be an algebra over the Hopf algebra A, and let M be an A-H Thom module. The *Wu map* of $M, \upsilon(\cdot, M): A \to H$, is the grade-preserving R-homomorphism defined inductively on $|a|$ by requiring

$$\upsilon(1, M) = 1 \quad \text{and} \quad w(a, M) = \sum_i a_i' \upsilon(a_i'', M).$$

Example 4.2. Let X be a closed n-dimensional manifold, τ the tangent bundle of X, and M the $\mathscr{A}_2$-$H^*(X; \mathbb{Z}/2)$ Thom module $\tilde{H}^{*+n}(T(\tau); \mathbb{Z}/2)$. Then by the classical Wu formula [8], $\upsilon(\mathrm{Sq}^i, M) = \upsilon_i(X)$, the ith Wu class of X.

Lemma 4.3. *For any A-H Thom module M,*

$$\upsilon(a, M) = \sum_i \chi(a_i') \cdot w(a_i'', M)$$

where $\chi: A \to A$ is the canonical conjugation.

Proof. We proceed by induction on $|a|$, the result being trivial if $|a| = 0$. Assume $|a| > 0$. By the inductive definition,

$$\upsilon(a, M) = w(a, M) - \sum_{|a_i''| < |a|} a_i' \cdot \upsilon(a_i'', M).$$

Write

$$\Delta a_i' = \sum_{j(i)} b_{j(i)}' \otimes b_{j(i)}'', \qquad \Delta a_i'' = \sum_{k(i)} c_{k(i)}' \otimes c_{k(i)}''.$$

By the inductive hypothesis,

$$\upsilon(a, M) = w(a, M) - \sum_{|a_i''| < |a|} a_i' \cdot \left[\sum_{k(i)} \chi(c_{k(i)}') \cdot w(c_{k(i)}'', M) \right]$$

$$= w(a, M) - \sum_{i,k(i)} a_i'\chi(c_{k(i)}') \cdot w(c_{k(i)}'', M) + \sum_i \chi(a_i') \cdot w(a_i'', M)$$

$$= w(a, M) - \sum_i \left[\sum_{j(i)} b_{j(i)}'\chi(b_{j(i)}'') \right] w(a_i'', M) + \sum_i \chi(a_i') \cdot w(a_i'', M),$$

this last equality following from the coassociativity of A. Since

$$\sum_{j(i)} b_{j(i)}'\chi(b_{j(i)}'') = \begin{cases} 0 & \text{if } |a_i'| > 0, \\ 1 & \text{if } a_i' = 1, \end{cases}$$

the assertion follows.

Lemma 4.4. *For every A-H Thom module M,*

$$\sum_i v(a_i', M) \cup v(a_i'', \bar{M}) = 0 \quad \textit{whenever } |a| > 0.$$

Proof. Writing $w = w(\cdot, M)$ and $\bar{w} = w(\cdot, \bar{M})$ we have, by Lemma 4.3,

$$\sum_i v(a_i', M) \cup v(a_i'', \bar{M}) = \sum_{i,j(i),k(i)} [\chi(b_{j(i)}') \cdot w(b_{j(i)}'')] \cup [\chi(c_{k(i)}') \cdot \bar{w}(c_{k(i)}'')]$$

where the notation is as in 4.3. By cocommutativity of A, this last expression equals

$$\text{(1)} \qquad \sum_{i,j(i),k(i)} (-1)^{|b_{j(i)}''||c_{k(i)}'|} [\chi(b_{j(i)}') \cdot w(c_{k(i)}')] \cup [\chi(b_{j(i)}'') \cdot \bar{w}(c_{k(i)}'')].$$

The cocommutativity of A implies χ is a morphism of coalgebras [4] from which it follows that the expression in (1) equals

$$\text{(2)} \qquad \sum_i \chi(a_i') \cdot \left[\sum_{k(i)} w(c_{k(i)}') \cup \bar{w}(c_{k(i)}'') \right].$$

Since

$$\sum_{k(i)} w(c_{k(i)}') \cup \bar{w}(c_{k(i)}'') = \begin{cases} 0 & \text{if } |a_i''| > 0, \\ 1 & \text{if } a_i'' = 1, \end{cases}$$

the expression in (2) equals $\chi(a) \cdot 1$, which is 0 since $|a| > 0$.

Theorem 4.5 (Wu Formula). *Let H be an algebra over the Hopf algebra A, M and A-H Thom module, and n a positive integer with the property that $a \cdot x = 0$ whenever $a \in A$, $x \in \bar{M}$, $|a| > 0$, and $|a| + |x| = n$. Then $a \cdot h = v(a, M) \cup h$ whenever $h \in H$, $a \in A$, and $|a| + |h| = n$.*

Proof. We proceed by induction on $|a|$, the conclusion being trivial when $|a| = 0$. Suppose $a \in A$, $h \in H$ satisfy $|a| > 0$ and $|a| + |h| = n$. Let $\bar{U}$ generate $\bar{M}^\circ$ as an H-module. By hypothesis, $a \cdot (h \cup \bar{U}) = 0$ which yields

$$\sum_i (-1)^{|a_i''||h|} (a_i' \cdot h) \cup w(a_i'', \bar{M}) = 0.$$

Thus, by Definition 4.1 we have, using the notation in the proof of Lemma 4.3,

$$\sum_{i,k(i)} (-1)^{|a_i''||h|}(a_i'\cdot h)\cup[c_{k(i)}'\cdot \upsilon(c_{k(i)}'',\bar{M})]=0.$$

By coassociativity of A and the Cartan formula,

$$\begin{aligned}0&=\sum_{i,j(i)} (-1)^{|h|(|b_{j(i)}''|+|a_i''|)}(b_{j(i)}'\cdot h)\cup[b_{j(i)}''\cdot \upsilon(a_i'',\bar{M})]\\&=\sum_{i} (-1)^{|a_i''||h|}a_i'\cdot[h\cup \upsilon(a_i'',\bar{M})]\\&=a\cdot h+\sum_{|a_i'|<|a|} a_i'\cdot[\upsilon(a_i'',\bar{M})\cup h].\end{aligned}$$

Thus, by the induction hypothesis and Lemma 4.4,

$$\begin{aligned}a\cdot h&=-\sum_{|a_i'|<|a|}\upsilon(a_i',M)\cup\upsilon(a_i'',\bar{M})\cup h\\&=\upsilon(a,M)\cup h-\left[\sum_i \upsilon(a_i',M)\cup\upsilon(a_i'',\bar{M})\right]\cup h\\&=\upsilon(a,M)\cup h.\end{aligned}$$

Example 4.6. Let X be a smooth connected closed n-dimensional manifold, smoothly embedded in the $n+k$ sphere with normal bundle ν. Suppose Y is a subcomplex of X of dimension $\leq n-2$ such that ν is trivial on Y. Then if $p: X\to X/Y$ denotes the projection, $\nu\cong p^*\xi$ for some k-plane bundle ξ over X/Y and $\tilde{H}^{n+k}(T(\xi);\mathbb{Z}/2)$ is generated by a spherical class. Thus the $\mathscr{A}_2$-$H^*(X/Y;\mathbb{Z}/2)$ Thom module $M=\tilde{H}^{*+k}(T(\xi);\mathbb{Z}/2)$ has the property that $a\cdot x=0$ whenever $a\in\mathscr{A}_2$, $x\in M$, $|a|>0$, and $|a|+|x|=n$. Thus, writing $\upsilon=\upsilon(\cdot,\bar{M})$, it follows from Theorem 4.5 that $a\cdot h=\upsilon(a)\cup h$ whenever $a\in\mathscr{A}_2$, $h\in H^*(X/Y;\mathbb{Z}/2)$, and $|a|+|h|=n$. The point of this trivial extension of the classical case is that $H^*(X/Y;\mathbb{Z}/2)$ need not satisfy Poincaré duality.

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